A singularity-free WEC-respecting time machine

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Abstract

A time machine (TM) is constructed whose creating in contrast to all TMs known so far requires neither singularities, nor violation of the weak energy condition (WEC). The spacetime exterior to the TM closely resembles the Friedmann universe.

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1 Introduction

This paper concerns an aspect of the long-standing question: How to create a time machine (or why is it impossible)? We begin with the following

Definition. Let N be an inextendible acausal spacetime. We call $L_N \subset N$ a *time machine* (created in the universe M) if

1. L_N comprises the causality violating set V:

$$L_N \supset V \equiv \{\mathcal{P} \mid \mathcal{P} \in N,\ J^+(\mathcal{P}) \cap J^-(\mathcal{P}) \neq \mathcal{P}\}$$

2. $N \setminus J^+(L_N)$ is isometric to $M \setminus J^+(L_M)$, where M is some inextendible causal spacetime and $\overline{L_M} \subset M$ is compact.

It is meant here that depending on whether we decide to make a time machine or not the world and our laborotory must be described by N and L_N, or by M and L_M, respectively. We require the compactness of L_M to differentiate TMs, which supposedly can be built by some advanced civilization, and causality violations of a cosmological nature such as the Gödel universe or the Gott "time machine" [1].

A few important facts are known about time machines. Among them:

1. Time machines with compact V (*compactly generated TMs, CTMs*) evolving from a noncompact partial Cauchy surface must violate WEC [2],

2. Creation of a CTM leads to singularity formation unless some energy condition (slightly stronger than WEC) is violated[1] [3].

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[1] Moreover, it is not clear whether singularities can be avoided at all, even at the sacrifice of WEC. Absence of singularities has not been proven, as far as I know, for any of TMs considered so far.

Though strictly speaking these facts do not *prove* that CTMs are impossible altogether, they, at least, can be used as starting point for the search for a mechanism protecting causality from CTMs [2].

CTMs, however, only constitute a specific class of TMs. Noncompactly generated TMs (NTMs) seem to be every bit as interesting as CTMs. Sometimes (see e. g. [2]) they are barred from consideration on the basis that some unpredictible information could enter noncompact V from a singularity or from infinity. This is so indeed, but compactness does not eliminate this trouble (in fact, compactness does not even eliminate singularities as a possible source of unpredictible information, as in the Misner space). The creation of a time machine is inherently connected with a loss of predictability (cf. [4]). One inevitably risks meeting something unexpected (i. e. not fixed by the data on the initial surface) as soon as one intersects a Cauchy horizon (by the very definition of the horizon). So, CTMs and NTMs are not too different in *this* regard.

It therefore seems important to find out whether there are any similar obstacles to creating NTMs. The example of the Deutsch-Politzer TM [5] showed that item 1 is not true in the case of NTMs. This TM, however, possesses singularities (and though mild, they are of such a nature that one cannot "smooth" them out [6]). Thus the following questions remained unanswered:

1. Do time machines without singularities exist?

2. Are the weak energy condition and the absence of singularities mutually exclusive for (noncompactly generated) time machines?

Our aim in this paper is to give the answers to both these questions (positive to the first and negative to the second). We make no attempt to discuss possible consequences of these answers. In particular, being interested only in the very existence of the desired TM we consider the fact that it may be created (see the Definition) in the spacetime resembling the Friedmann universe, only as a pleasant surprise.

2 Construction of the time machine

To construct a singularity-free TM it would be natural to start from the Deutsch-Politzer TM and to look for an appropriate conformal transformation of its metric which would move the dangerous points to infinity. However, it is not easy in the four-dimensional case to find a transformation (if it exists) yielding both WEC fulfilment and b- (or BA-) completeness. So, we shall use somewhat different means [7]. First, by a conformal transformation we make a part of the *two-dimensional* Deutsch-Politzer spacetime locally complete (the spacetime Q_N below), then compose L_N obeying WEC from Q_N and some S (chosen so that it does not spoil the completeness), and finally embed the resulting TM in an appropriate N.

2.1 Curved Deutsch-Politzer time machine Q_N

Let Q_f be a square q_m

$$q_m \equiv \{\chi, \tau \,\big|\, m > |\chi|, |\tau|\} \tag{1}$$

endowed with the following metric:

$$ds^2 = f^{-2}(\tau, \chi)(-d\tau^2 + d\chi^2) \tag{2}$$

Here f is a smooth bounded function defined on $\mathbb{R}^2 \supset q_m$ such that

$$f(\tau, \chi) = 0 \quad \Leftrightarrow \quad \{\tau = \pm h, \ \chi = \pm h\} \tag{3}$$

with $0 < h < m/2$. The four points $f^{-1}(0)$ bound two segments

$$l_\pm \equiv \{\chi, \tau \,|\, \tau = \pm h, \ |\chi| < h\}$$

and we require that

$$f(\tau, \chi)|_{U^+} = f(\tau - 2h, \chi), \tag{4}$$

where U^+ is some neighborhood of l^+.

Now (as is done with the Minkowski plane in the case of the "usual" Deutsch-Politzer spacetime [5]) remove the points $f^{-1}(0)$ from q_m, make cuts along l_+ and l_-, and glue the upper bank of each cut with the lower bank of the other (see Fig. 1). The resulting spacetime Q_N ("curved Deutsch-Politzer TM") is

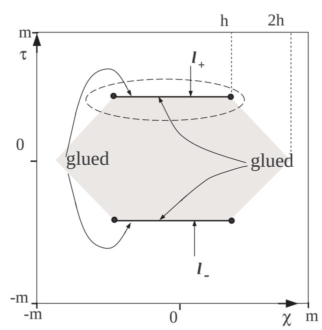

Figure 1: Curved Deutsch-Politzer time machine. The shaded region is the causality violating set. The dashed line bounds U^+.

not, of course, diffeomorphic to Q_f. Nevertheless, for simplicity of notation we shall continue to use the "old coordinates" τ, χ for its points.

2.2 The time machine L_N

Let S be a two-sphere with the standard metric:

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5}$$

where R is a constant satisfying (to bring about WEC fulfillment, see below)

$$R^{-2} \geqslant \max_{\overline{q}_m} \big(f(f,_{\chi\chi} - f,_{\tau\tau}) + f,_\tau^2 - f,_\chi^2 \big) \tag{6}$$

Then

$$L_N \equiv Q_N \times S \tag{7}$$

is just the desired time machine.

2.3 Exemplary spacetimes N, M

To find an appropriate N require in addition to (3,4,6)

$$f(\mathcal{P}) = const \equiv f_0 \quad \text{when } \mathcal{P} \notin q_m \tag{8}$$

and denote by $\widetilde{Q}$ the spacetime obtained by replacing $q_m \to \mathbb{R}^2$ in the definition of Q_N. It follows from what is proven in the next section that the spacetime $\widetilde{Q} \times S$ is inextendible and could thus be taken as N (with, for example, $L_M \equiv Q_{f_0} \times S$). We would like, however, to construct another, more "realistic" N.

Consider a manifold $\mathbb{R}^1 \times \mathrm{S}^3$ with the metric

$$ds^2 = a^2[-d\tau^2 + d\chi^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{9}$$

Here τ is a coordinate on $\mathbb{R}^1$ and θ, φ, $\chi\,(-\pi/2 \leqslant \chi \leqslant \pi/2)$ are polar coordinates on S^3. Impose the following conditions on a, ρ (it suffices to choose $m < \pi/2,\ f_0 < R^{-1}\cos m$ for their feasibility):

$$\text{on } q_m \qquad a = 1/f, \qquad \rho = fR \tag{10a}$$

$$\text{exterior to } q_m \qquad a = \hat{a}(\tau), \qquad \rho = \hat{\rho}(\chi), \tag{10b}$$

where $\hat{\rho}$, $\hat{a}$ are *convex* positive functions and for some $n \in (m, \pi/2)$ holds $\hat{\rho}|_{|\chi|>n} = \cos\chi$.

It is easy to see that the region $|\tau|$, $|\chi| < m$ of this manifold is $Q_f \times S$. So

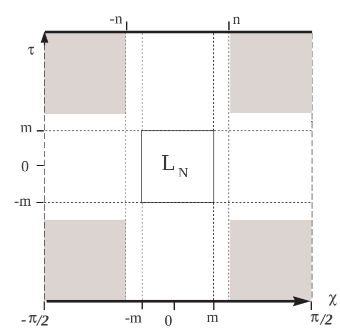

Figure 2: "Almost Friedmann" time machine. Shaded regions are parts of the Friedmann universe. The thick horizontal lines depict cosmological singularities.

we can repeat the manipulations with cuts and obtain a TM with the metric (9) on $N \setminus \overline{L_N}$ (see Fig. 2), that is the TM is created in a spacetime with the metric of the Friedmann universe outside some spherical layer and some time interval.

3 Proofs

3.1 Weak energy condition

The metric of the time machine L_N due to (10a) is given by (2,5) and the condition (6) guarantees that the weak (and even the dominant) energy conditions hold there (see [7] for details).

In the outer space $N \setminus L_N$ the metric is given by (9). Introducing the quantities

$$\Phi \equiv \Lambda \equiv \ln a, \quad r \equiv a\rho$$

we bring it to the form (14.49) of ref. [8]. The fact that by (10b) a and ρ are positive and convex gives us:

$$\ddot{\Phi} \leqslant 0, \quad \rho''/\rho \leqslant 0, \quad 1 - {\rho'}^2 \geqslant 0, \tag{11}$$

(in the last inequality we have also used that $\rho'(\pm\pi/2) = \mp 1$.) Hence (see [8] for notation)

$$\begin{aligned} &E = \bar{E} = a^{-2}\ddot{\Phi} \leqslant 0, \quad H = 0, \\ &F = r^{-2}(1 - {\rho'}^2) + a^{-2}\dot{\Phi}^2 \geqslant 0, \\ &\bar{F} = a^{-2}(\dot{\Phi}^2 - \rho''/\rho) \geqslant 0 \end{aligned}$$

So (see (14.52) of ref. [8]), WEC holds in this region too.

3.2 Completeness

The results of [7] prove that there are no "BA-singularities" in L_N, that is any timelike inextendible (in N) curve $\gamma \subset L_N$ with bounded acceleration has infinite proper length. There is a popular idea, however, that only b-complete regions may be accepted as singularity-free. So, the remainder of the article is devoted to the proof of the fact that L_N (not the whole N, where cosmological singularities $\hat{a} = 0$ present) has no "b-singularities." We shall use the following new notation:

$$\begin{gathered} x^1 \equiv \tau, \quad x^2 \equiv \chi, \quad x^3 \equiv \theta, \quad x^4 \equiv \varphi, \\ \alpha \equiv \chi + \tau, \quad \beta \equiv \chi - \tau, \quad \dot{} \equiv d/ds. \end{gathered}$$

Let $\gamma(s) = x^i(s)$, $i = 1, \ldots 4$ be a C^1 curve in L_N. It defines two other curves (its projections onto Q_N and S):

$$Q_N \supset \quad \gamma_Q(s) \equiv x^k(s), \qquad k = 1, 2 \tag{12}$$

$$S \supset \quad \gamma_S(s) \equiv x^j(s), \qquad j = 3, 4 \tag{13}$$

Lying in L_N and Q_N the curves γ and γ_Q can be considered at the same time as lying in N and $\widetilde{Q}$, respectively. We shall call such curves inextendible if they are inextendible in those "larger" spacetimes. Let $\{\mathbf{e}_{(i)}(s)\}$ be an orthonormal basis in the point $\gamma(s)$, obtained from $\{\mathbf{e}_{(i)}(0)\}$ by parallel propagating along γ and $\mathbf{e}_{(\alpha)} \equiv \mathbf{e}_{(1)} + \mathbf{e}_{(2)}$, $\mathbf{e}_{(\beta)} \equiv \mathbf{e}_{(1)} - \mathbf{e}_{(2)}$. Choosing $\mathbf{e}_{(i)}(0) \sim \partial_{x^i}$ and solving the equations $\nabla_{\dot{\gamma}}\mathbf{e}_{(i)} = 0$ one immediately finds

$$\nabla_{\dot{\gamma}_Q}\mathbf{e}_{(k)} = 0, \quad \nabla_{\dot{\gamma}_S}\mathbf{e}_{(j)} = 0 \tag{14}$$

and

$$e^i_{(\mu)}(s) = e^i_{(\mu)}(0) \exp\{2\int_0^s \phi_{,\mu}\,\dot{\mu}\,ds'\} \quad (\mu \equiv \alpha, \beta) \tag{15}$$

where $\phi \equiv \ln f$.

The "affine length" μ_N of γ is by definition [9]

$$\mu_N[\gamma] \equiv \int_0^1 \Big(\sum_i \langle\dot\gamma, \mathbf{e}_{(i)}\rangle^2\Big)^{1/2} ds \qquad (16)$$

We define affine lengths $\mu_Q[\gamma_Q]$ and $\mu_S[\gamma_S]$ by changing i in (16) to k and j, respectively. Due to (14) these definitions are consistent. Obviously,

$$\mu_N[\gamma] \geqslant 1/2\,(\mu_Q[\gamma_Q] + \mu_S[\gamma_S]) \qquad (17)$$

Proposition. If $\gamma \subset L_N$ is inextendible then $\mu_N[\gamma] = \infty$.

If γ is inextendible, than either γ_S or γ_Q (or both) are inextendible, too. But S is obviously b-complete. So the Proposition follows from (17) coupled with the following

Lemma. If γ_Q is inextendible, then $\mu_Q[\gamma_Q] = \infty$.

Let us introduce a function Φ on γ_Q:

$$\Phi \equiv \int_0^s (\phi_{,\alpha}\,\dot\alpha - \phi_{,\beta}\,\dot\beta)\,ds'$$

and let us split γ_Q on segments $\gamma_n \equiv \gamma[s_n, s_{n+1})$ so that the sign of Φ does not change on γ_n:

$$\gamma_Q = \bigcup_n \gamma_n, \quad \gamma_n : \Phi(s_n) = 0,\ \Phi(s_n < s < s_{n+1}) \leqslant 0 \text{ (or } \geqslant 0). \qquad (18)$$

Denote the contribution of a segment γ_n in $\mu_Q[\gamma_Q]$ by μ_n. Since

$$\big(\langle\dot\gamma, \mathbf{e}_{(1)}\rangle^2 + \langle\dot\gamma, \mathbf{e}_{(2)}\rangle^2\big)^{1/2} \geqslant 1/2\,\big(|\langle\dot\gamma, \mathbf{e}_{(\alpha)}\rangle| + |\langle\dot\gamma, \mathbf{e}_{(\beta)}\rangle|\big)$$

we can write for μ_n (cf. (15)):

$$\mu_n \geqslant C_1 \int\limits_{s_n}^{s_{n+1}} f^{-2}\Big(|\dot\alpha| \exp\{2\int_0^s \phi_{,\beta}\,\dot\beta\,ds'\} + |\dot\beta| \exp\{2\int_0^s \phi_{,\alpha}\,\dot\alpha\,ds'\}\Big)\,ds \qquad (19)$$

Here and subsequently we denote by $C_p,\ p = 1,\ldots$ some irrelevant positive constants factored out from the integrand. Using

$$\big[f(s)\big]^{-2} = \big[f(0)\big]^{-2} \exp\{-2\int_0^s (\phi_{,\alpha}\,\dot\alpha + \phi_{,\beta}\,\dot\beta)\,ds'\} \qquad (20)$$

we can rewrite (19) as

$$\mu_n \geqslant C_2 \int\limits_{s_n}^{s_{n+1}} \Big(|\dot\alpha| \exp\{-2\int_0^s \phi_{,\alpha}\,\dot\alpha\,ds'\} + |\dot\beta| \exp\{-2\int_0^s \phi_{,\beta}\,\dot\beta\,ds'\}\Big)\,ds \qquad (21)$$

For definiteness let $\Phi \leqslant 0$ on γ_n. Then the first exponent in (21) is greater than the second and we can replace it by their geometric mean, that is [see (20)] by

$f(0)/f(s)$. So,

$$\mu_n \geqslant C_3 \int_{s_n}^{s_{n+1}} |\dot{\alpha}/f|\, ds \geqslant C_4 \int_{s_n}^{s_{n+1}} |\phi_{,\alpha}\, \dot{\alpha}|\, ds$$
$$\geqslant -C_5 \int_{s_n}^{s_{n+1}} (\phi_{,\alpha}\, \dot{\alpha} + \phi_{,\beta}\, \dot{\beta})\, ds \geqslant C_5 \big(\phi(s_n) - \phi(s_{n+1})\big) \quad (22)$$

(The third inequality follows again from $\Phi \leqslant 0$.) Clearly, (22) also holds for those γ_n where $\Phi \geqslant 0$ and hence summing over n gives:

$$\mu_Q[\gamma_Q] \geqslant C_5 \big(\phi(0) - \phi(s)\big), \quad \forall s \in [0, 1) \quad (23)$$

There are no closed null geodesics in $\widetilde{Q}$. So, it is "locally complete" [10]. That is any inextendible γ_Q either has $\mu_Q[\gamma_Q] = \infty$, or leaves any compact subset of $\widetilde{Q}$. But in the latter case (recall that $\gamma_Q \in Q_N$) $\phi(s)$ is unbounded below, which due to (23) again gives

$$\mu_Q[\gamma_Q] = \infty.$$

□

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